Rational parametrization of conchoids to algebraic curves

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Abstract We study the rationality of each of the components of the conchoid to an irreducible algebraic affine plane curve, excluding the trivial cases of the lines through the focus and the circle centered at the focus and radius the distance involved in the conchoid. We prove that conchoids having all their components rational can only be generated by rational curves. Moreover, we show that reducible conchoids to rational curves have always their two components rational. In addition, we prove that the rationality of the conchoid component, to a rational curve, does depend on the base curve and on the focus but not on the distance. As a consequence, we provide an algorithm that analyzes the rationality of all the components of the conchoid and, in the affirmative case, parametrizes them. The algorithm only uses a proper parametrization of the base curve and the focus and, hence, does not require the previous computation of the conchoid. As a corollary, we show that the conchoid to the irreducible conics, with conchoid-focus on the conic, are rational and we give parametrizations. In particular we parametrize the *Limaçons of Pascal*. We also parametrize the conchoids of *Nicomedes*. Finally, we show how to find the foci from where the conchoid is rational or with two rational components.

Keywords Conchoid curve · Rational parametrization

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1 Introduction

The conchoid is a classical geometric construction. Intuitively speaking, if $\mathcal{C}$ is a plane curve (the base curve), A a fixed point in the plane (the focus), and d a non-zero fixed field element (the distance), the conchoid to $\mathcal{C}$ from the focus A at distance d is the (closure of) set of points Q in the line AP at distance d of a point P varying in the curve $\mathcal{C}$. The two classical and most famous conchoids are the Conchoid of *Nicomedes* ($\mathcal{C}$ is a line and $A \notin \mathcal{C}$) and the *Limaçons of Pascal* ($\mathcal{C}$ is a circle and $A \in \mathcal{C}$). Conchoids are useful in many applications as construction of buildings, astronomy, electromagnetic research, physics, optics, engineering in medicine and biology, mechanical in fluid processing, etc (see the introduction of [3] for references).

In [3] we define formally the concept of conchoid of a plane curve by means of incidence diagrams, and we presented a theoretical analysis of the main properties of conchoids to irreducible curves (see Sect. 2 for a brief summary). In [1], the notion of conchoid curve is generalized to the concept of conchoidal transformation of two curves and its main properties are established; when one of the two curves is a circle the conchoidal transformation turns to be the classical conchoid curve. In addition, among other results, in [1], the authors prove that the conchoid of a generic curve is irreducible, and they present a genus formula for this generic case.

In this paper, we also deal with the genus of the conchoid (indeed with the genus zero problem), but without the assumption of generality on the original base curve. More precisely, given an irreducible plane curve, we deal with the problem of analyzing the rationality of all (note that the conchoid might not be irreducible) the components of the conchoid and, in the affirmative case, the actual computation of rational parametrizations for each rational component. Clearly this problem can be approached by computing the implicit equation of the conchoid to afterwards factor it to finally apply to each factor any parametrization algorithm. Nevertheless, we want to avoid all these computations solving the problem directly from the input base curve and the focus. For this purpose, similarly as in [3], we work over an algebraically closed field $\mathbb{K}$ of characteristic zero, and curves are considered reduced; that is, they are the zero set in $\mathbb{K}^2$ of non-constant square-free polynomials of $\mathbb{K}[y_1, y_2]$. Furthermore, if a curve is defined by the square-free polynomial f, when we speak about its components, we mean the curves defined by the non-constant irreducible factors (over $\mathbb{K}$) of f (see [4] for further details). In addition, when we refer to the defining polynomial of a plane curve we are meaning the square-free polynomial generating its ideal.

In the theoretical analysis presented in [3], three different types of curves have an exceptional behavior: the isotropic lines $(y_1 - a) \pm \sqrt{-1}\,(y_2 - b) = 0$ where $A = (a, b)$ is the focus (their conchoid is empty), the circle centered at the focus and radius the distance involved in the conchoid (its conchoid has a zero-dimensional component) and the lines through the focus (all conchoid components are special; see Sect. 2 for this concept). Note that the first case is, in fact, included in the third. For all the other cases, the most remarkable property in [3] is that the conchoid is a plane algebraic curve with at most two component, at least one of them being simple (see Sect. 2 for the notion of simple component).

In this paper, we exclude w.l.o.g. the above three exceptional types of curves. In this situation, we prove that conchoids having all their components rational can only be generated by rational curves. Moreover, we show that reducible conchoids to rational curves have always their two components rational; we call this case **double rationality**. Furthermore, we characterize rational conchoids and double rational conchoids. From these results, one deduces that the rationality of the conchoid component, to a rational curve, does depend on the base curve and on the focus but not on the distance. To approach the problem we use similar ideas to those in [2] introducing the notion of **reparametrization curve** (see Definition 2) as well as the notion of RDF parametrization (see Definition 1). The RDF concept allows us to detect the double rationality while the reparametrization curve is a much simpler curve than the conchoid, directly computed from the input rational curve and the focus, and that behaves equivalently as the conchoid in terms of rationality. As a consequence of these theoretical results we provide an algorithm to solve the problem. Given a proper parametrization of the base curve and the focus, the algorithm analyzes the rationality of all the components of the conchoid and, in the affirmative case, parametrizes them. We note that the algorithm does not require the computation of the conchoid. In addition, we show that the conchoid to the irreducible conics, with conchoid-focus on the conic, are rational and we give parametrizations. In particular we parametrize the *Limaçons of Pascal*. We also parametrize the conchoids of *Nicomedes*. Finally, we show how to find the foci from where the conchoid is rational or with two rational components.

2 Preliminaries on conchoids and general assumptions

In this section we recall the notion of conchoid as well as its main properties. For further details, we refer to [3]. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. In $\mathbb{K}^2$ we consider the symmetric bilinear form

$$\mathfrak{b}((x_1, x_2), (y_1, y_2)) = x_1 y_1 + x_2 y_2,$$

which induces a metric vector space with light cone of isotropy $\mathfrak{L} = \{P \in \mathbb{K}^2 \mid \mathfrak{b}(P, P) = 0\}$ (see [5]). That is, $\mathfrak{L}$ is the union of the two lines defined by $x_1 \pm \sqrt{-1}\, x_2 = 0$. In this context, the circle of center $P \in \mathbb{K}^2$ and radius $d \in \mathbb{K}$ is the plane curve defined by $\mathfrak{b}(\bar{x} - P, \bar{x} - P) = d^2$, with $\bar{x} = (x_1, x_2)$. We say that the distance between $P, Q \in \mathbb{K}^2$ is $d \in \mathbb{K}$ if P is on the circle of center Q and radius d. The notion of "distance" is hence defined up to multiplication by ± 1. On the other hand, if $P \in \mathbb{K}^2$ is not isotropic (i.e. $P \notin \mathfrak{L}$) we denote by $\|P\|$ any of the elements in $\mathbb{K}$ such that $\|P\|^2 = \mathfrak{b}(P, P)$, and if $P \in \mathbb{K}^2$ is isotropic, then $\|P\| = 0$. In this paper we usually work with both solutions of $\|P\|^2 = \mathfrak{b}(P, P)$. For this reason we use the notation $\pm\|P\|$.

In this situation, let $\mathcal{C}$ be the affine irreducible plane curve defined by the irreducible polynomial $f(\bar{y}) \in \mathbb{K}[\bar{y}]$, $\bar{y} = (y_1, y_2)$, let $d \in \mathbb{K}^*$ be a non-zero field element, and let $A = (a, b) \in \mathbb{K}^2$. We consider the **(conchoid) incidence variety**

$$\mathfrak{B}(\mathcal{C}) = \left\{ (\bar{x}, \bar{y}, \lambda) \in \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \;\middle/\; \begin{array}{l} f(\bar{y}) = 0 \\ \|\bar{x} - \bar{y}\|^2 = d^2 \\ \bar{x} = A + \lambda(\bar{y} - A) \end{array} \right\}$$

and the incidence diagram

$$\begin{tikzcd}[column sep=small]
& \mathfrak{B}(\mathcal{C}) \subset \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\pi_1(\mathfrak{B}(\mathcal{C})) \subset \mathbb{K}^2 & & \mathcal{C} \subset \mathbb{K}^2
\end{tikzcd}$$

where

$$\begin{array}{llll} \pi_1 : \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \longrightarrow \mathbb{K}^2, & & \pi_2 : \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \longrightarrow \mathbb{K}^2 \\ \quad (\bar{x}, \bar{y}, \lambda) \quad \longmapsto \bar{x} & & \quad (\bar{x}, \bar{y}, \lambda) \quad \longmapsto \bar{y}. \end{array}$$

Then, we define the conchoid to $\mathcal{C}$ from the focus A and distance d as the algebraic Zariski closure in $\mathbb{K}^2$ of $\pi_1(\mathfrak{B}(\mathcal{C}))$, and we denote it by $\mathfrak{C}(\mathcal{C})$; i.e.

$$\mathfrak{C}(\mathcal{C}) = \overline{\pi_1(\mathfrak{B}(\mathcal{C}))}.$$

For details on how to compute the conchoid see [3]. In general, A and d are just precise elements in $\mathbb{K}^2$ and $\mathbb{K}^*$, respectively. When this will not be the case (for instance in Sect. 5) the conchoid will be denoted by $\mathfrak{C}(\mathcal{C}, A, d)$ instead of $\mathfrak{C}(\mathcal{C})$ to emphasize this fact.

Throughout this paper, we assume w.l.o.g. that:

1. $\mathcal{C}$ is none of the isotropic lines passing through the focus $(y_1 - a) \pm \sqrt{-1}(y_2 - b) = 0$. This ensures that $\mathfrak{C}(\mathcal{C}) \neq \emptyset$.
2. $\mathcal{C}$ is not a circle centered at A and radius d. If $\mathcal{C}$ is such a circle, then $\mathfrak{C}(\mathcal{C})$ decomposes as the focus union the circle centered at A and radius $2d$. This assumption avoids that the conchoid has zero-dimensional components (compare to Theorem 1).
3. $\mathcal{C}$ is not a line through the focus. If $\mathcal{C}$ is such a line, then $\mathfrak{C}(\mathcal{C}) = \mathcal{C}$. This assumption avoids that the conchoid has all components special (compare to Theorem 2).

The following theorem (see Theorem 1 in [3]) states the main property on conchoids.

Theorem 1 *$\mathfrak{C}(\mathcal{C})$ has at most two components and all of them have dimension 1.*

Now, we recall the notion of simple and special components of a conchoid which, as shown in [3], play an important role when studying the rationality. More precisely, an irreducible component $\mathcal{M}$ of $\mathfrak{C}(\mathcal{C})$ is called simple if there exists a non-empty Zariski dense subset $\Omega \subset \mathcal{M}$ such that, for $Q \in \Omega$, $\mathrm{Card}(\pi_2(\pi_1^{-1}(Q))) = 1$. Otherwise $\mathcal{M}$ is called special. The next theorem states the main property on the existence of simple components (see Theorem 3 in [3]).

Theorem 2 *$\mathfrak{C}(\mathcal{C})$ has at least one simple component.*

The next lemma (see Lemma 5 in [3]) connects the birationality of the maps in incidence diagram and the simple components of the conchoid.

Lemma 1 *Let π_1, π_2 be the projections in the incidence diagram of $\mathcal{C}$, and $\mathcal{M}$ an irreducible component of $\mathfrak{C}(\mathcal{C})$.*

(1) *If $\mathfrak{C}(\mathcal{C})$ is reducible, the restricted map $\pi_2|_{\pi_1^{-1}(\mathcal{M})} : \pi_1^{-1}(\mathcal{M}) \longrightarrow \mathcal{C}$ is birational.*

(2) *The restricted map $\pi_1|_{\pi_1^{-1}(\mathcal{M})} : \pi_1^{-1}(\mathcal{M}) \longrightarrow \mathcal{M}$ is birational iff $\mathcal{M}$ is simple.*

3 Rational conchoids

We know that conchoids are either irreducible or with two components (see Theorem 1). In this section, we characterize the conchoids having all their components rational. We see that these conchoids can only be generated by rational curves. Moreover, we characterize the cases where the conchoid is rational or it is reducible with the two components rational. As a consequence, we prove that the conchoid to the irreducible conics, from a focus on the conic, are rational; in particular all *Limaçons of Pascal* are rational. We also see that all conchoids of *Nicomedes* are rational.

Let $\mathcal{P}(t)$ be a rational parametrization of $\mathcal{C}$, and let

$$\Omega = \mathcal{P}(\mathbb{K}) \setminus (\{A\} \cup \mathcal{L}^{+} \cup \mathcal{L}^{-})$$

where $\mathcal{L}^{\pm}$ are the isotropic lines through the focus. By hypothesis $\mathcal{C}$ is none of the lines $\mathcal{L}^{\pm}$, and hence Ω is a non-empty Zariski open subset of $\mathcal{C}$. Also let $\Sigma \subset \mathbb{K}$ be the subset of parameter values generating the elements in Ω.

Then $\pi_2^{-1}(\Omega)$ is the set

$$\left\{ \left(\mathcal{P}(t_0) + \frac{d}{\pm\|\mathcal{P}(t_0) - A\|}(\mathcal{P}(t_0) - A), \mathcal{P}(t_0), 1 + \frac{d}{\pm\|\mathcal{P}(t_0) - A\|} \right) \Big/ \ t_0 \in \Sigma \right\}.$$

So, by Lemma 3(1) in [3],

$$\pi_1(\pi_2^{-1}(\Omega)) = \left\{ \mathcal{P}(t_0) + \frac{d}{\pm\|\mathcal{P}(t_0) - A\|}(\mathcal{P}(t_0) - A) \Big/ \ t_0 \in \Sigma \right\}$$

is a non-empty Zariski open subset of $\mathfrak{C}(\mathcal{C})$. Thus, if $\pm\|\mathcal{P}(t) - A\| \in \mathbb{K}(t)$, $\mathcal{P}(t) + \frac{d}{\pm\|\mathcal{P}(t)-A\|}(\mathcal{P}(t) - A)$ parametrizes all the components of $\mathfrak{C}(\mathcal{C})$. This motivates the next definition.

Definition 1 We say that a parametrization $\mathcal{P}(t) \in \mathbb{K}(t)^2$ is at rational distance to the focus if $\|\mathcal{P}(t) - A\|^2 = m(t)^2$, with $m(t) \in \mathbb{K}(t)$. For short, we express this fact saying that $\mathcal{P}(t)$ is RDF or A-RDF if we need to specify the focus.

Remark 1 Note that:

1. As one can see in Sect. 5, the notion of RDF depends on the focus; see for instance Examples 6 and 10.

2. If $\mathcal{P}(t)$ is A-RDF, every re-parametrization of $\mathcal{P}(t)$ is also A-RDF. However, if can happen that a re-parametrization of a non RDF parametrization is RDF. For instance, as we have seen above (t, t^2) is not $(0, 0)$-RDF but $\left(\frac{2t}{t^2-1}, \frac{4t^2}{(t^2-1)^2}\right)$ is $(0, 0)$-RDF since

$$\left(\frac{2t}{t^2-1}\right)^2 + \left(\frac{2t}{t^2-1}\right)^4 = \frac{4t^2(t^2+1)^2}{(t^2-1)^4}.$$

   So, we have that if $\mathcal{C}$ has a proper RDF parametrization then all the parametrizations of $\mathcal{C}$ are RDF with respect to the same focus. Nevertheless, it might happen that $\mathcal{C}$ does not have proper RDF parametrizations but has non-proper RDF parametrizations. We recall that a parametrization $\mathcal{P}(t)$ is said proper if the induced rational map $\mathcal{P} : \mathbb{K} \to \mathcal{C}, t \mapsto \mathcal{P}(t)$ is birational or equivalently if $\mathbb{K}(\mathcal{P}(t)) = \mathbb{K}(t)$ (see [4], Sect. 4.2, for further details).

Checking whether a given parametrization is RDF is easy. However deciding, and actually computing, the existence of RDF reparametrizations of non RDF parametrizations is not so direct. For dealing with this, we introduce the notion of reparametrizing curve whose importance will appear clear in the following theorems. The motivation for the origin of this concept can be found in the proof of (1) $\Rightarrow$ (3) of Theorem 3. More precisely: we want that $\|\mathcal{P}(t) - A\|$ is a rational function, say $m(t)$. This implies that $1/m(\mathcal{P}(t) - A)$ parametrizes the unit circle centered at the origin, and hence it should be a reparametrization of any the proper parametrization of the circle, for instance of $(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1})$. Now a simple algebraic manipulation of this fact provides the implicit equation defining the reparametrizing curve.

Definition 2 Let $\mathcal{P}(t) \in \mathbb{K}(t)^2$ be a rational parametrization of $\mathcal{C}$. We define the **reparametrizing curve of** $\mathcal{P}(t)$, and we denote it by $\mathfrak{G}(\mathcal{P})$, as the curve whose defining polynomial is the primitive part with respect to x_2 of the numerator of $\mathfrak{b}((-2x_2, x_2^2 - 1), \mathcal{P}(x_1) - A)$.

Remark 2 We observe that:

1. $\mathfrak{G}(\mathcal{P})$ does not depend on the representatives of the rational functions in $\mathcal{P}(t)$.
2. Let H be the primitive part w.r.t. x_2 of the numerator of $\mathfrak{b}((-2x_2, x_2^2-1), \mathcal{P}(x_1) - A)$. Since $\mathcal{C}$ is not a line through the focus, H can be written as $H = \alpha(x_1)x_2^2 + \beta(x_1)x_2 - \alpha(x_1)$ where α, β are not identically zero; in particular H has degree 2 in x_2. Furthermore, since $\mathcal{C}$ is none of the isotropic lines passing through the focus, it also holds that H is square-free. Therefore the defining polynomial of $\mathfrak{G}(\mathcal{P})$, namely H, has degree 2 w.r.t. x_2 and it is primitive w.r.t. x_2. So, if $\mathfrak{G}(\mathcal{P})$ is reducible then it has two factors, both depending linearly on x_2.
3. Let $\mathcal{P}(t), \mathcal{Q}(t)$ be parametrizations of $\mathcal{C}$, and $\varphi(t) \in \mathbb{K}(t)$ such that $\mathcal{Q}(t) = \mathcal{P}(\varphi(t))$. Let

$$M_1 = \mathfrak{b}((-2x_2, x_2^2 - 1), \mathcal{P}(x_1) - A), \quad M_2 = \mathfrak{b}((-2x_2, x_2^2 - 1), \mathcal{Q}(x_1) - A).$$

   Then, $M_1(\varphi(x_1), x_2) = M_2(x_1, x_2)$. □

The following theorem characterizes the conchoids, having all the components rational, by means of the notions of RDF and reparametrizing curve. In fact, we show that conchoids having all their components rational can only be generated by rational curves; indeed iff the base curve is rational and has RDF parametrizations.

Theorem 3 *The following statements are equivalent:*

(1) *$\mathcal{C}$ is rational and has an* RDF *parametrization.*
(2) *$\mathfrak{C}(\mathcal{C})$ has at least one rational simple component.*
(3) *There exists a proper parametrization of $\mathcal{C}$ whose reparametrizing curve has at least one rational component.*
(4) *The reparametrizing curve of every proper parametrization of $\mathcal{C}$ has at least one rational component.*
(5) *All the components of $\mathfrak{C}(\mathcal{C})$ are rational.*

Proof We prove that all the statements are equivalent to (1). To prove that (2) implies (1), let $\mathcal{M}$ be a rational simple component of $\mathfrak{C}(\mathcal{C})$ parametrized by $\mathcal{R}(t) = (R_1(t), R_2(t))$. We consider the diagram:

$$\begin{tikzcd}[column sep=small]
 & \Gamma = \pi_1^{-1}(\mathcal{M}) \subset \mathfrak{B}(\mathcal{C}) \subset \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \arrow[dl, "\tilde{\pi}_1 = \pi_1|_{\pi_1^{-1}(\mathcal{M})}"'] \arrow[dr, "\tilde{\pi}_2 = \pi_2|_{\pi_1^{-1}(\mathcal{M})}"] & \\
\mathcal{M} \subset \mathfrak{C}(\mathcal{C}) \subset \mathbb{K}^2 & & \mathcal{C} \subset \mathbb{K}^2 \\
\mathbb{K} \arrow[u, "\mathcal{R}"'] & &
\end{tikzcd}$$

Since $\mathcal{M}$ is simple, by Lemma 1, $\tilde{\pi}_1$ is birational. So, $\mathcal{Q}(t) = \tilde{\pi}_2(\tilde{\pi}_1^{-1}(\mathcal{R}(t)))$ parametrizes $\mathcal{C}$. Let us see that $\mathcal{Q}(t) := (Q_1(t), Q_2(t))$ is RDF. By construction, $\tilde{\pi}_1^{-1}(\mathcal{R}(t)) = (\mathcal{R}(t), \mathcal{Q}(t), \lambda) \in \mathfrak{B}(\mathcal{C})$, where $\lambda = (R_1 - a)/(Q_1 - a) = (R_2 - b)/(Q_2 - b)$ and $\|\mathcal{R}(t) - \mathcal{Q}(t)\|^2 = d^2$. Note that $\mathcal{C}$ is not a line passing through A, and hence $Q_1 \neq a$, $Q_2 \neq b$. Moreover, $\lambda \neq 1$, since otherwise $\mathcal{R}(t) = \mathcal{Q}(t)$ that yields to $d = 0$. So $\mathcal{Q}(t) - A = (\mathcal{Q}(t) - \mathcal{R}(t)) + (\mathcal{R}(t) - A) = (\mathcal{Q}(t) - \mathcal{R}(t)) + \lambda(\mathcal{Q}(t) - A)$, and hence $\|\mathcal{Q}(t) - A\|^2 = d^2/(\lambda - 1)^2$.

In order to prove that (1) implies (2), let $\mathcal{P}(t)$ be an RDF parametrization of $\mathcal{C}$. Let $\|\mathcal{P}(t) - A\|^2 = m(t)^2$. Then

$$\left(\mathcal{P}(t) \pm \frac{d}{m(t)}(\mathcal{P}(t) - A), \mathcal{P}(t), 1 \pm \frac{d}{m(t)}\right) \in \mathfrak{B}(\mathcal{C}).$$

Moreover, since $\mathcal{P}(t)$ generates a dense subset of $\mathcal{C}$, by Lemma 3 in [3], $\mathcal{P}(t) \pm \frac{d}{m(t)}(\mathcal{P}(t) - A)$ generates a dense in $\mathfrak{C}(\mathcal{C})$. So, all components of $\mathfrak{C}(\mathcal{C})$ are rational. Now, the result follows from Theorem 2.

To see that (1) implies (3), let $\mathcal{P}(t) = (P_1(t), P_2(t))$ be an RDF parametrization of $\mathcal{C}$, and $\|\mathcal{P}(t) - A\|^2 = m(t)^2$. Then, $1/m(t)(\mathcal{P}(t) - A)$ parametrizes the circle $x_1^2 + x_2^2 = 1$. Since $\mathcal{R}(t) = (\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1})$ is a proper parametrization of the circle, it holds that there exists $\phi(t) \in \mathbb{K}(t)$ such that $\mathcal{R}(\phi(t)) = 1/m(t)(\mathcal{P}(t) - A)$. This implies that $\mathfrak{b}((-2\phi(t), \phi(t)^2 - 1), \mathcal{P}(t) - A) = 0$. Therefore $(t, \phi(t))$ parametrizes one component of $\mathfrak{G}(\mathcal{P})$.

To prove that (3) implies (1), let $(\phi_1(t), \phi_2(t))$ be a parametrization of one component of $\mathfrak{G}(\mathcal{P})$, where $\mathcal{P}(t) = (P_1(t), P_2(t))$ is a proper parametrization of $\mathcal{C}$. Then, $\mathfrak{b}((-2\phi_2(t), \phi_2(t)^2 - 1), \mathcal{P}(\phi_1(t)) - A) = 0$. Note that ϕ_2 is not identically zero since otherwise it would imply that $P_2(\phi_1) = b$ and $\mathcal{C}$ is not a line passing through the focus. Then, it follows that $\mathcal{P}(\phi_1(t))$ is RDF; indeed

$$\|\mathcal{P}(\phi_1(t)) - A\|^2 = \frac{(\phi_2(t)^2 + 1)^2}{(2\phi_2(t))^2}(P_2(\phi_1(t)) - b)^2.$$

Trivially (4) implies (3). In order to prove that (3) implies (4), let $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ be two proper parametrizations of $\mathcal{C}$, such that $\mathfrak{G}(\mathcal{P})$ has at least one rational component. Let $(\phi_1(t), \phi_2(t))$ be a parametrization of one component of $\mathfrak{G}(\mathcal{P})$. Note that, because of Remark 2 (2), $\phi_1(t)$ is not constant. Since both parametrizations are proper, there exists an invertible $\varphi \in \mathbb{K}(t)$ such that $\mathcal{Q}(t) = \mathcal{P}(\varphi(t))$. Let $M_1(x_1, x_2) = \mathfrak{b}((-2x_2, x_2^2 - 1), \mathcal{P}(x_1) - A)$ and $M_2(x_1, x_2) = \mathfrak{b}((-2x_2, x_2^2 - 1), \mathcal{Q}(x_1) - A)$. Let D_i be the denominator of M_i and let C_i, H_i be, respectively, the content and primitive part w.r.t. x_2 of the numerator of M_i. Then, by Remark 2 (3),

$$C_1(x_1)H_1(x_1, x_2)D_2(\varphi^{-1}(x_1)) = D_1(x_1)C_2(\varphi^{-1}(x_1))H_2(\varphi^{-1}(x_1), x_2).$$

So, $D_1(\phi_1)C_2(\varphi^{-1}(\phi_1))H_2(\varphi^{-1}(\phi_1), \phi_2) = 0$. Since $\phi_1 \notin \mathbb{K}$, then $\varphi^{-1}(\phi_1(t)) \notin \mathbb{K}$. Since D_1, C_2 are non-zero univariate polynomials, $D_1(\phi_1)C_2(\varphi^{-1}(\phi_1)) \neq 0$. Therefore, $H_2(\varphi^{-1}(\phi_1), \phi_2) = 0$. Hence, $(\varphi^{-1}(\phi_1), \phi_2)$ parametrizes a component of $\mathfrak{G}(\mathcal{Q})$. Therefore one concludes (4). For the implication of (1) implies (5) see the proof of (1) implies (2). Furthermore, if (5) holds, then $\mathfrak{C}(\mathcal{C})$ has at least one rational simple component, and by (2) one concludes (1). □

Remark 3 Theorem 3 implies that:

1. Conchoids with all their components rational can only be generated by rational curves.
2. The rationality of all the components of the conchoid does depend on the base curve and the focus, but not on the distance.

Corollary 1 *The conchoid to a rational curve is either rational, or it is reducible with two rational components, or it is irreducible but non-rational.*

Proof Let $\mathcal{C}$ be the base curve, and let $\mathfrak{C}(\mathcal{C})$ be reducible. Then, by Corollary 3 in [3] and Theorem 2, at least one conchoid simple component is rational. Now the corollary follows from Theorem 3. □

Remark 4 If $\mathcal{C}$ is non-rational, it might happen that its conchoid is reducible with two non-rational components or with one component non-rational and the other rational. For instance, if $\mathcal{C}$ is the curve defined by the polynomial $f(y_1, y_2) = 81 + 162y_2^2y_1^6 + 1521y_1^4y_2^2 + 972y_1y_2^4 + 162y_1^2y_2^4 - 1944y_1^3y_2^3 - 1944y_1^5y2 + 864y_1^3y_2^2 - 2898y_1^2y_2^3 -$

$8730y_1^4y_2 - 1404y_1^2 - 1458y_2 + 108y_1 - 324y_1^4y_2^3 + 81y_2^4y_1^4 - 17388y_1^3y_2 + 972y_1^5y_2^2 - 162y_2^5y_1^2 - 162y_1^6y_2 - 13122y_1^2y_2 + 6480y_2^2 - 162y_2^3 - 972y_2y_1 + 8694y_2^2y_1 + 4203y_1^2y_2^2 + 1449y_2^4 - 1932y_1^3 + 5590y_1^4 + 972y_1^7 + 81y_1^8 + 8532y_1^5 + 4356y_1^6 + 81y_2^6$
then the conchoid to C from the focus $A = (-3, 0)$ and distance $d = 1/3$, has two components defined by the factors
$(-x_2 + x_1^2)\,(1296 + 1728x_1 - 5832x_2 + 6237x_2^2 - 3888x_2 * x_1 + 3690x_1^2x_2^2 - 4968x_1^2 - 13122x_1^2x_2 - 7584x_1^3 + 2689x_1^4 - 162x_1^6x_2 + 1422x_2^4 + 972x_1^7 - 162x_2^5x_1^2 + 972x_1^5x_2^2 - 17064x_1^3x_2 - 648x_2^3 + 81x_2^4x_1^4 - 8676x_1^4x_2 + 8532x_1x_2^2 + 81x_1^8 - 2844x_1^2x_2^3 + 7884x1^5 + 540x_1^3x_2^2 - 1944x_1^5x_2 + 4302x_1^6 - 1944x_1^3x_2^3 + 162x_1^2x_2^4 + 972x_1x_2^4 + 1467x_1^4x_2^2 + 162x_2^2x_1^6 + 81x_2^6 - 324x_1^4x_2^3)$,
one of them is the parabola (rational) and the other is a non-rational curve of genus 1. Note that the reason is that simple components of reducible conchoids are birationally equivalent to C (see Corollary 3 in [3]). Indeed, the parabola is an special component of the conchoid. This fact justifies that the original curve C is, in fact, the conchoid to the parabola taking the same focus and distance (see Theorem 2 in [3]). Also, take into account that for almost all values of d the conchoid has all the components simple (see Theorem 4 in [3]).

Corollary 2 *Let $\mathcal{P}(t)$ be a parametrization of C such that $\mathfrak{G}(\mathcal{P})$ has at least one rational component $\mathcal{M}$, and let $(\phi_1(t), \phi_2(t))$ be a parametrization of $\mathcal{M}$. Then $\mathcal{P}(\phi_1(t))$ is* RDF.

Proof It follows from the proof of (3) implies (1) in Theorem 3. □

Corollary 3 *Let $\mathcal{P}(t)$ be a proper parametrization of C. Then, the following statements are equivalent:*

(1) *All the components of $\mathfrak{C}(C)$ are rational.*
(2) *There exists $\varphi \in \mathbb{K}(t)$ of degree at most two such that $\mathcal{P}(\varphi(t))$ is* RDF.

Proof (2) implies (1) follows from Theorem 3. (1) implies (2) follows from Theorem 3, from Corollary 2, and using that the partial degree of $\mathfrak{G}(\mathcal{P})$ w.r.t. x_2 is 2 (see Theorem 4.21 in [4]). □

In the sequel, we analyze the case of conchoids to rational curves, characterizing the rational conchoid and conchoids with two rational components; we refer to this case as **double rationality**.

Lemma 2 *$\mathfrak{G}(\mathcal{P})$ is reducible if and only if $\mathcal{P}(t)$ is* RDF.

Proof Let H be the primitive part w.r.t. x_2, and $M(x_1)$ the content w.r.t. x_2 of the numerator of $\mathfrak{b}((-2x_2, x_2^2 - 1), \mathcal{P}(x_1) - A)$. Note that H is the defining polynomial of $\mathfrak{G}(\mathcal{P})$ and has degree 2 w.r.t. x_2 (see Remark 2 (2)). All factors of H depend on x_2. Thus, $\mathfrak{G}(\mathcal{P})$ is reducible if and only if H has two factors depending on x_2, or equivalently, the discriminant Δ_H w.r.t. x_2 is the square of a polynomial. Therefore, since $M(x_1)^2\Delta_H = 4\,\|\mathcal{P}(t) - A\|^2$, one has that $\mathfrak{G}(\mathcal{P})$ is reducible if and only if $\mathcal{P}(t)$ is RDF. □

Theorem 4 (Characterization of double rational conchoids) *Let C be rational. The following statement are equivalent:*

(1) *$\mathfrak{C}(C)$ is reducible.*
(2) *$\mathfrak{C}(C)$ has exactly two components and they are rational.*
(3) *There exists an* RDF *proper parametrization of C.*
(4) *Every proper parametrization of C is* RDF.
(5) *There exists a proper parametrization of C whose reparametrizing curve is reducible.*
(6) *The reparametrizing curve of every proper parametrization of C is reducible.*

Proof By Corollary 1, (1) implies (2). (2) implies (1) trivially. In order to prove that (2) implies (3), let $\mathcal{R}(t)$ be a proper parametrization of a simple component $\mathcal{M}$ of $\mathfrak{C}(C)$. We consider the diagram used in the proof of Theorem 3. By Lemma 1, $\tilde{\pi}_2 \circ \tilde{\pi}_1^{-1} \circ \mathcal{R} : \mathbb{K} \longrightarrow C$ is birational. Therefore, $\mathcal{Q}(t) = \pi_2(\pi_1^{-1}(\mathcal{R}(t)))$ is a proper parametrization of C. Furthermore, reasoning as in the proof of "(2) implies (1)", in Theorem 3, one has that $\mathcal{Q}(t)$ is RDF. (3) implies (4) follows from Remark 1.

In order to see that (4) implies (2) we observe that, because of Lüroth's Theorem, (4) implies (3). Now, let $\mathcal{P}(t) = (P_1(t), P_2(t))$ be an RDF proper parametrization of C. Reasoning as in the proof of Theorem 3, we get that

$$\mathcal{R}^{\pm}(t) = (R_1^{\pm}(t), R_2^{\pm}(t)) := \mathcal{P}(t) \pm d\, \frac{\mathcal{P}(t) - A}{\|\mathcal{P}(t) - A\|}$$

parametrizes all components of $\mathfrak{C}(C)$. So, it only remains to prove that $\mathfrak{C}(C)$ is reducible. Let us assume that it is irreducible. Then, by Theorem 2, $\mathfrak{C}(C)$ is simple, and, by Lemma 1 (2), $\mathfrak{B}(C)$ is irreducible. Moreover,

$$\mathcal{M}^{\pm}(t) = \left(\mathcal{R}^{\pm}(t), \mathcal{P}(t), \frac{R_1^{\pm}(t) - a}{P_1(t) - a} \right)$$

are two rational parametrizations of $\mathfrak{B}(C)$. Moreover, since $\mathcal{P}(t)$ is proper then $\mathbb{K}(t) = \mathbb{K}(\mathcal{P}(t)) \subset \mathbb{K}(\mathcal{M}^{\pm}(t)) \subset \mathbb{K}(t)$. So, each $\mathcal{M}^{\pm}(t)$ is proper. Therefore, there exists a linear rational function $\varphi(t)$ such that $\mathcal{M}^{+}(\varphi(t)) = \mathcal{M}^{-}(t)$. Thus, $\varphi(t) = t$ and, since $d \neq 0$, $\mathcal{P}(t) = A$ which is a contradiction.

Applying Lemma 2 one has that (4) implies (5). The implication "(5) implies (6)" follows from Lemma 2 and Remark 1. Finally, "(6) implies (4)" follows directly from Lemma 2. □

Theorem 5 (Characterization of rational conchoids) *Let C be rational. The following statement are equivalent:*

(1) *$\mathfrak{C}(C)$ is rational (and hence irreducible).*
(2) *There exists a proper parametrization of C whose reparametrizing curve is rational.*
(3) *The reparametrizing curve of every proper parametrization of C is rational.*

Proof Let $\mathfrak{C}(\mathcal{C})$ be rational. By Theorem 3, there exists a proper parametrization $\mathcal{P}(t)$ of $\mathcal{C}$ such that $\mathfrak{G}(\mathcal{P})$ has at least one rational component; say $\mathcal{M}$. Furthermore, by Theorem 4, $\mathcal{P}(t)$ is not RDF. Thus, by Lemma 2, $\mathfrak{G}(\mathcal{P})$ is irreducible, and hence rational. So, (1) implies (2).

We prove that (2) implies (3). Let $\mathcal{P}(t)$ proper such that $\mathfrak{G}(\mathcal{P})$ is rational and let $\mathcal{Q}(t)$ be another proper parametrization of $\mathcal{C}$. Since $\mathfrak{G}(\mathcal{P})$ is irreducible, by Lemma 2, $\mathcal{P}(t)$ is not RDF. By Remark 1 $\mathcal{Q}(t)$ is not RDF. So, by Lemma 2, $\mathfrak{G}(\mathcal{Q})$ is irreducible. Now, the result follows from Theorem 3.

Finally, we prove that (3) implies (1). Let $\mathfrak{G}(\mathcal{P})$ be rational, with $\mathcal{P}(t)$ proper. By Lemma 2, $\mathcal{P}(t)$ is not RDF. Thus, by Theorem 4, $\mathfrak{C}(\mathcal{C})$ is irreducible. Now, the result follows from Theorem 3. □

We apply these results to the case of conchoids to conics with the focus on the conic (in particular to *Limaçons of Pascal*), and to the case of conchoids of *Nicomedes*.

Lemma 3 *Let* $\mathcal{P}(t) = (p_1(t)/p(t), p_2(t)/p(t))$ *be a proper parametrization of* $\mathcal{C}$ *with* $\gcd(p_1, p_2, p) = 1$ *and* $\deg_t(p_i/p) \leq 2$ *for* $i = 1, 2$. *If* $A \in \mathcal{C}$, *then* $\mathfrak{C}(\mathcal{C})$ *is rational.*

Proof The defining polynomial $g(x_1, x_2)$ of $\mathfrak{G}(\mathcal{P})$ is the primitive part w.r.t. x_2 of $K(x_1, x_2) = -2x_2(p_1(x_1) - ap(x_1)) + (x_2^2 - 1)(p_2(x_1) - bp(x_1))$. Moreover, the content $C(x_1)$ of K w.r.t. x_2 is $\gcd(p_1(x_1) - ap(x_1), p_2(x_1) - bp(x_1))$. First, we observe that $\deg_{x_1}(g) > 0$. Indeed, if it is zero, it implies that there exist $\lambda, \mu \in \mathbb{K}$ such that $\mathcal{P}(t) = (a + \lambda C(t)/p(t), b + \mu C(t)/p(t))$ and, hence, $\mathcal{C}$ would be a line passing through the focus.

Let us assume that $A = (a, b)$ is reachable by $\mathcal{P}(t)$; say $\mathcal{P}(t_0) = A$. Then $x_1 - t_0$ divides $C(x_1)$, and hence $\deg_{x_1}(g) = 1$. So $\mathfrak{G}(\mathcal{P})$ is rational and, by Theorem 5, $\mathfrak{C}(\mathcal{C})$ is rational. Now, if A is not reachable by $\mathcal{P}(t)$, then for $i = 1, 2$, $\deg(p_i) \leq \deg(p)$ (see Sect. 6.3. in [4]). Say

$$p_i(x_1) = a_{i,n}x_1^n + \cdots + a_{i,0}, \quad p(x_1) = b_n x_1^n + \cdots + b_0,$$

where $a_{i,n}$ might vanish. Then, $A = (a_{1,n}/b_n, a_{2,n}/b_n)$ (see Sect. 6.3. in [4]). So, $\deg_{x_1}(g) = 1$. Thus, reasoning as above we get the result. □

Now, taking into account the parametrizations of the irreducible conics, by Lemma 3, one deduces the following result (see also Examples 1, 2, 3 and 4).

Corollary 4 *Let* $\mathcal{C}$ *be an irreducible conic, and let* $A \in \mathcal{C}$, *then* $\mathfrak{C}(\mathcal{C})$ *is rational.*

Corollary 5 *Limaçons of Pascal are rational.*

Remark 5 In general it is not true that if the focus is on the curve, the conchoid is rational. For instance, let $\mathcal{C}$ be the curve defined by $y_1^3 y_2 = 1$, $\mathcal{P}(t) = (1/t, t^3)$, and $A = (1, 1) \in \mathcal{C}$. Then, $\mathfrak{G}(\mathcal{P})$ is defined by $x_2^2x_1^3 - x_1^3 + x_1^2x_2^2 - x_1^2 + x_1x_2^2 - x_1 + 2x_2$ and its genus is 2.

Finally we analyze the conchoids of *Nicomedes* (see also Example 5).

Corollary 6 *Conchoids of Nicomedes are rational.*

Proof Conchoids of *Nicomedes* appear when $\mathcal{C}$ is a line and $A \notin \mathcal{C}$. Let $\mathcal{P}(t) = (p_1(t), p_2(t)) = (a_1+\lambda_1 t, a_2+\lambda_2 t)$. The defining polynomial of $\mathfrak{G}(\mathcal{P})$ is $g(x_1, x_2) = -2x_2(p_1(x_1)-a)+(x_2^2-1)(p_2(x_1)-b)$. Note that, since $A \notin \mathcal{C}$, g is primitive w.r.t. x_2. Now the result follows from Theorem 5 and noting that $\deg_{x_1}(g) = 1$. □

4 Parametrization of conchoids

In this section we apply the results in Sect. 3 to derive an algorithm to check the rationality of the components of a conchoid and, in the affirmative case, to parametrize them. For this purpose, in the sequel, let $\mathcal{C}$ be rational and $\mathcal{P}(t)$ be a proper parametrization of $\mathcal{C}$. We also assume that the focus A is fixed. However, we consider d generic. Recall that we have assume that $\mathcal{C}$ is not a line through the focus neither a circle centered at the focus and radius d; nevertheless, observe that the problem for these two excluded cases is trivial.

First, we check whether $\mathcal{P}(t)$ is RDF; equivalently one can check whether $\mathfrak{G}(\mathcal{P})$ is reducible. If so, by Theorem 4, $\mathfrak{C}(\mathcal{C})$ is double rational and

$$\mathcal{P}(t) + \frac{d}{\pm\|\mathcal{P}(t) - A\|}(\mathcal{P}(t) - A)$$

parametrizes the two components. If $\mathcal{P}(t)$ is not RDF, we check whether $\mathfrak{G}(\mathcal{P})$ is rational. If it is not rational, by Theorem 5, $\mathfrak{C}(\mathcal{C})$ is not rational. If $\mathfrak{G}(\mathcal{P})$ is rational, by Theorem 5, $\mathfrak{C}(\mathcal{C})$ is rational. In order to parametrize $\mathfrak{C}(\mathcal{C})$, we get a proper parametrization $(\phi_1(t), \phi_2(t))$ of $\mathfrak{G}(\mathcal{P})$ (see [4] for this). Then, by Corollary 2, $\mathcal{Q}(t) = \mathcal{P}(\phi_1(t))$ is RDF. Therefore, any of the parametrizations

$$\mathcal{Q}(t) + \frac{d}{\pm\|\mathcal{Q}(t) - A\|}(\mathcal{Q}(t) - A)$$

paramatrizes $\mathfrak{C}(\mathcal{C})$. Summarizing we get the following procedure:

1. Compute the primitive part $g(x_1, x_2)$ w.r.t. x_2 of the numerator of $\mathfrak{b}((-2x_2, x_2^2-1), \mathcal{P}(x_1) - A)$.
2. If g is reducible **return** that $\mathfrak{C}(\mathcal{C})$ is double rational and that $\mathcal{P}(t) + \frac{d}{\pm\|\mathcal{P}(t)-A\|}(\mathcal{P}(t) - A)$ parametrize the two components.
3. Check whether the genus of $\mathfrak{G}(\mathcal{P})$ is zero. If not **return** that $\mathfrak{C}(\mathcal{C})$ is not rational.
4. Compute a proper parametrization $(\phi_1(t), \phi_2(t))$ of $\mathfrak{G}(\mathcal{P})$ and **return** that $\mathfrak{C}(\mathcal{C})$ is rational and that $\mathcal{P}(\phi_1(t)) + \frac{d}{\pm\|\mathcal{P}(\phi_1(t))-A\|}(\mathcal{P}(\phi_1(t)) - A)$ parametrizes $\mathfrak{C}(\mathcal{C})$.

We illustrate the algorithm by means of some examples.

Example 1 (Conchoid to Parabolas) Let $\mathcal{C}$ be the parabola defined by $f(y_1, y_2) = y_2 - \mu_1 y_1^2 + \mu_2 y_1 + \mu_3$, with $\mu_1 \neq 0$. We consider the proper parametrization $\mathcal{P}(t) = (t, \mu_1 t^2 + \mu_2 t + \mu_3)$, and the focus $A = (\lambda, \mu_1\lambda^2 + \mu_2\lambda + \mu_3)$ being any point on $\mathcal{C}$. By Corollary 4, we know that $\mathfrak{C}(\mathcal{C})$ is rational. Here, we indeed compute a parametrization. The polynomial g defining $\mathfrak{G}(\mathcal{P})$ is irreducible:

$$g(x_1, x_2) = \mu_1 x_1 x_2^2 - \mu_1 x_1 + \mu_2 x_2^2 - \mu_2 - 2x_2 + \lambda\mu_1 x_2^2 - \lambda\mu_1.$$

Moreover $\mathfrak{G}(\mathcal{P})$ is rational and can be parametrized as (recall that $\mu_1 \neq 0$)

$$\phi(t) = (\phi_1(t), \phi_2(t)) = \left(-\frac{t^2\mu_2 - \mu_2 - 2t + \lambda\, t^2\mu_1 - \lambda\, \mu_1}{\mu_1\,(t^2-1)}, t\right).$$

Therefore, $\mathcal{Q}(t) = (\phi_1(t), \mu_1\phi_1(t)^2 + \mu_2\phi_1(t) + \mu_3)$ is RDF and $\mathcal{Q}(t) + \frac{d}{\pm\|\mathcal{Q}(t)-A\|}(\mathcal{Q}(t) - A)$ parametrizes $\mathfrak{C}(\mathcal{C})$.

Example 2 **(Conchoid to Ellipses)** Let $\mathcal{C}$ be the ellipse defined by

$$f(y_1, y_2) = \frac{y_1^2}{r_1^2} + \frac{y_2^2}{r_2^2} - 1,$$

with $r_1 r_2 \neq 0$. We consider the proper parametrization

$$\mathcal{P}(t) = \left(\frac{2\,r_1 t}{t^2+1}, \frac{r_2\,(t^2-1)}{t^2+1}\right),$$

and the focus $A = \mathcal{P}(\lambda)$ being a point on $\mathcal{C}$. By Corollary 4, we know that $\mathfrak{C}(\mathcal{C})$ is rational. Here, we indeed compute a parametrization. The polynomial g, defining $\mathfrak{G}(\mathcal{P})$, is irreducible:

$$g(x_1, x_2) = 2\,x_1 x_2 r_1 \lambda + x_1 r_2 {x_2}^2 - r_2 x_1 - 2\,x_2 r_1 + \lambda\, r_2 {x_2}^2 - r_2\lambda.$$

Moreover $\mathfrak{G}(\mathcal{P})$ is rational and can be parametrized as

$$\phi(t) = (\phi_1(t), \phi_2(t)) = \left(-\frac{-2\,r_1 t + \lambda\, r_2 t^2 - r_2\lambda}{2\,t r_1\lambda + r_2 t^2 - r_2}, t\right).$$

Therefore, $\mathcal{Q}(t) = \mathcal{P}(\phi_1(t))$ is RDF and $\mathcal{Q}(t) + \frac{d}{\pm\|\mathcal{Q}(t)-A\|}(\mathcal{Q}(t) - A)$ parametrizes $\mathfrak{C}(\mathcal{C})$.

Example 3 **(Limaçon of Pascal)** Taking $r_1 = r_2 \neq 0$ in Example 2, we get a parametrization of the *Limaçons of Pascal*.

Example 4 **(Conchoid to Hyperbolas)** Let $\mathcal{C}$ be the hyperbola defined by

$$f(y_1, y_2) = \frac{y_1^2}{r_1^2} - \frac{y_2^2}{r_2^2} - 1,$$

with $r_1 r_2 \neq 0$. We consider the proper parametrization

$$\mathcal{P}(t) = \left(-\frac{r_1\left(r_1^2 + r_2^2 t^2\right)}{-r_1^2 + r_2^2 t^2}, \frac{2 r_2^2 r_1 t}{-r_1^2 + r_2^2 t^2} \right),$$

and the focus $A = \mathcal{P}(\lambda)$ being a point on $\mathcal{C}$. By Corollary 4, we know that $\mathfrak{C}(\mathcal{C})$ is rational. Here, we indeed compute a parametrization. The polynomial g, defining $\mathfrak{G}(\mathcal{P})$, is irreducible:

$$g(x_1, x_2) = -x_1 x_2^2 \lambda\, r_2^2 - 2\, x_1 x_2 r_1^2 + \lambda\, r_2^2 x_1 - x_2^2 r_1^2 + r_1^2 - 2\, \lambda\, x_2 r_1^2.$$

Moreover $\mathfrak{G}(\mathcal{P})$ is rational and can be parametrized as

$$\phi(t) = (\phi_1(t), \phi_2(t)) = \left(-\frac{r_1^2\left(t^2 - 1 + 2\,\lambda\, t\right)}{\lambda\, r_2^2 t^2 + 2\, r_1^2 t - \lambda\, r_2^2}, t \right).$$

Therefore, $\mathcal{Q}(t) = \mathcal{P}(\phi_1(t))$ is RDF and $\mathcal{Q}(t) + \frac{d}{\pm\|\mathcal{Q}(t) - A\|}(\mathcal{Q}(t) - A)$ parametrizes $\mathfrak{C}(\mathcal{C})$.

Example 5 (Conchoid of Nicomedes) Let $\mathcal{C}$ be the line parametrized by

$$\mathcal{P}(t) = (a_1 + t\lambda_1, a_2 + t\lambda_2),$$

and the focus $A = (a, b) \notin \mathcal{C}$. Then, $\mathfrak{C}(\mathcal{C})$ is the conchoid of *Nicomedes*. By Corollary 4, we know that $\mathfrak{C}(\mathcal{C})$ is rational. Here, we indeed compute a parametrization. The polynomial g, defining $\mathfrak{G}(\mathcal{P})$, is irreducible because $A \notin \mathcal{C}$:

$$g(x_1, x_2) = -2x_2 a_1 - 2x_2 x_1 \lambda_1 + 2x_2 a + x_2^2 a_2 + x_2^2 x_1 \lambda_2 - x_2^2 b - a_2 - x_1 \lambda_2 + b.$$

Moreover $\mathfrak{G}(\mathcal{P})$ is rational and can be parametrized as

$$\phi(t) = (\phi_1(t), \phi_2(t)) = \left(\frac{2ta_1 + t^2 b - 2ta - t^2 a_2 - b + a_2}{-2t\lambda_1 - \lambda_2 + t^2 \lambda_2}, t \right).$$

Therefore, $\mathcal{Q}(t) = \mathcal{P}(\phi_1(t))$ is RDF and $\mathcal{Q}(t) + \frac{d}{\pm\|\mathcal{Q}(t) - A\|}(\mathcal{Q}(t) - A)$ parametrizes $\mathfrak{C}(\mathcal{C})$.

5 Detecting foci to parametrize conchoids

In the previous section we have seen how to decide whether the conchoid to a rational curve is rational or double rational and, in the affirmative case, how to parametrize the components of the conchoid. Nevertheless, in that reasoning the focus is fixed.

In this section, we analyze a slightly different problem. We assume that we are given a proper parametrization

$$\mathcal{P}(t) = \left(\frac{p_1(t)}{p(t)}, \frac{p_2(t)}{p(t)} \right),$$

where $\gcd(p_1, p_2, p) = 1$, of a rational curve $\mathcal{C}$. We also assume that the smallest subfield $\mathbb{L}$ of $\mathbb{K}$ containing the coefficients of $\mathcal{P}(t)$ is computable. Then, we look for $A_0 \in \mathbb{K}^2$ such that the conchoid $\mathfrak{C}(\mathcal{C}, A_0, d)$ has all its components rational. We know that this implies that either $\mathfrak{C}(\mathcal{C}, A_0, d)$ has two rational components or it is rational. In the first case we say that the A_0 is a **double rational focus** and, in the second, that A_0 is a **rational focus**. For this purpose, in the sequel, $A = (a, b)$ is treated generically, and hence a, b are unknowns.

For this purpose, we first analyze the case where $\mathcal{C}$ is a line. Afterwards study the general case, where we will distinguish between double rational foci and rational foci detection.

The case of lines. Let $\mathcal{C}$ be a line. We can assume w.l.o.g. that $\mathcal{C}$ is the line defined by $y_2 = 0$. Then, the double rational foci are those such that $\|(t, 0) - (a, b)\|^2$ is the square of a rational function. That is, those such that $(t - a)^2 + b^2$ is the square of a linear polynomial in t; i.e. $b = 0$. Therefore, the double rational foci seem to be all points on $\mathcal{C}$. Note that this case corresponds to the unique degenerated situation where the conchoid is irreducible and special (see Corollary 2 in [3]).

Finally we observe that, in Corollary 6, we have already analyzed the problem of detecting rational foci for lines, and we have seen that for all foci, not on the line, the conchoid is rational.

Detecting double rational foci. The strategy is as follows. First we determine a set $\mathcal{F}$ in $\mathbb{K}^2$ containing the possible double rational foci. Afterwards, we prove that $\mathcal{F}$ is the union of $\mathcal{C}$ and finitely many lines. So all components of $\mathcal{F}$ are rational, and using a parametrization of each component we determine conditions on the parameter to get double rational foci.

We now assume that $\mathcal{C}$ is not a line, an we proceed as follows. Let

$$\Delta_1(a, t) = p_1(t) - a\, p(t), \;\; \Delta_2(b, t) = p_2(t) - b\, p(t),$$
$$\tilde{\Sigma}_1(a, b, t) = \Delta_1 + \sqrt{-1}\,\Delta_2, \;\; \tilde{\Sigma}_2(a, b, t) = \Delta_1 - \sqrt{-1}\,\Delta_2,$$

where these polynomials are regarded as polynomials in $\mathbb{K}[a, b, t]$.

Lemma 4 *It holds that*

1. $\gcd(\tilde{\Sigma}_1, \tilde{\Sigma}_2) = 1$.
2. *For every* $(a_0, b_0) \in \mathbb{K}^2$, $\tilde{\Sigma}_1(a_0, b_0, t)\tilde{\Sigma}_2(a_0, b_0, t)$ *is not constant.*

Proof First note that there exist $\mu_{i,j} \in \mathbb{K}$ such that $\Delta_i = \mu_{i,1}\tilde{\Sigma}_1 + \mu_{i,2}\tilde{\Sigma}_2$, for $i = 1, 2$.

(1) If W is a factor of $\gcd(\tilde{\Sigma}_1, \tilde{\Sigma}_2)$, then W divides Δ_1 and Δ_2. Since W divides Δ_1, then $W \in \mathbb{K}[a, t]$. Since W divides Δ_2, then $W \in \mathbb{K}[b, t]$. So, $W \in \mathbb{K}[t]$. However, this implies that W divides $\gcd(p_1, p_2, p) = 1$. Therefore, $W \in \mathbb{K}$.

(2) Let $a_0, b_0 \in \mathbb{K}$ be such that $\tilde{\Sigma}_1(a_0, b_0, t)\tilde{\Sigma}_2(a_0, b_0, t) \in \mathbb{K}$. Then it holds that either there exists $i \in \{1, 2\}$ such that $\tilde{\Sigma}_i(a_0, b_0, t) = 0$ (say, $i = 1$) or both $\tilde{\Sigma}_i(a_0, b_0, t)$, $i = 1, 2$, are constant. In the first case, $\Delta_1(a_0, b_0, t) = -\sqrt{-1}\Delta_2(a_0, b_0, t)$ and therefore $p_1(t)/p(t) + \sqrt{-1}p_2(t)/p(t) = a + \sqrt{-1}b$. So, C is a line and this case has been excluded. In the second case, one deduces that $\Delta_1(a_0, t), \Delta_2(b_0, t) \in \mathbb{K}$. Say $\Delta_1(a_0, t) = \mu_1$, $\Delta_2(b_0, t) = \mu_2$, with $\mu_1, \mu_2 \in \mathbb{K}$. However, this implies that C is the line $\mu_2(y_1 - a_0) = \mu_1(y_2 - b_0)$, and this case has been excluded. □

Now, let the content $C_i(t)$ of $\tilde{\Sigma}_i$, w.r.t. $\{a, b\}$, factor over $\mathbb{K}$ as

$$C_i(t) = \alpha_i(t)^2 \beta_i(t),$$

where $\gcd(\alpha_i, \beta_i) = 1$ and β_i is square-free. Then, we introduce the new polynomials

$$\Sigma_i = \frac{\tilde{\Sigma}_i}{\alpha_i(t)^2}, \text{ and } \Sigma = \Sigma_1\Sigma_2.$$

So,

$$\|\mathcal{P}(t) - A\|^2 = \frac{\Delta_1^2 + \Delta_2^2}{p(t)^2} = \frac{\tilde{\Sigma}_1\tilde{\Sigma}_2}{p(t)^2} = \left(\frac{\alpha_1(t)\alpha_2(t)}{p(t)}\right)^2 \Sigma_1\Sigma_2 = \left(\frac{\alpha_1(t)\alpha_2(t)}{p(t)}\right)^2 \Sigma.$$

Thus, a necessary condition for $A_0 = (a_0, b_0) \in \mathbb{K}^2$ to be a double rational focus of $\mathcal{P}(t)$ is that $\Sigma(a_0, b_0, t)$ has multiple roots, regarded as a polynomial in the indeterminate t.

We now prove that $\Sigma \notin \mathbb{K}[a, b]$. Indeed, if Σ does not depend on t, then $\Sigma_1 \in \mathbb{K}[a, b]$. Therefore, $\alpha_1^2(t)\Sigma_1(a, b) = \tilde{\Sigma}_1 = \Delta_1(t) + \sqrt{-1}\Delta_2(t) = (p_1 + \sqrt{-1}\, p_2) - (a + \sqrt{-1}\, b)p$. So, p and $p_1 + \sqrt{-1}\, p_2$ are multiple of $\alpha_1(t)^2$. Hence, $\mathcal{P}(t)$ parametrizes a line of the type $x + \sqrt{-1}y = \lambda$, and we have excluded the case of lines.

Then, let $R(a, b)$ be the square-free part of the resultant $\mathrm{Res}_t(\Sigma, \partial\Sigma/\partial t)$ of $\Sigma, \partial\Sigma/\partial t$ w.r.t. t. Note that, since Σ does depend on t, R is well defined. In this situation, the double rational foci belong to the algebraic set $\mathcal{F}$ in $\mathbb{K}^2$ defined by $R(a, b) = 0$. Nevertheless, we must ensure that $\mathcal{F}$ is not the whole plane, and hence that the condition is not trivial.

Lemma 5 *$R(a, b)$ is not identically zero.*

Proof If $R = 0$, then Σ has a repeated factor. By Lemma 4 (1), $\gcd(\Sigma_1, \Sigma_2)$ is also 1. So, at least one of the two polynomials Σ_1, Σ_2 has a repeated factor, say Σ_1. Since the total degree in $\{a, b\}$ of Σ_1 is 1, the repeated factor belongs to $\mathbb{K}[t]$. But this impossible because of the construction of Σ_1. □

Finally, we analyze the structure of $\mathcal{F}$. For this purpose, we denote by $\mathrm{Lc}_v(g)$ the leading coefficient of a polynomial g w.r.t. the variable v.

Lemma 6 *Let $g_1, g_2 \in \mathbb{U}[t]$ and $g = g_1 g_2$, where $\mathbb{U}$ is a unique factorization domain of characteristic zero. There exists $\epsilon \in \mathbb{N}$ such that*

$$\mathrm{Res}_t(g, g') = (-1)^{\epsilon} \mathrm{Res}_t(g_1, {g_1}') \mathrm{Res}_t(g_2, {g_2}') \mathrm{Res}_t(g_1, g_2)^2,$$

where h' denotes the derivative w.r.t. t of $h \in U[t]$.

Proof The next equalities are given up to multiplication by (-1). First, $\mathrm{Res}_t(g, g') = \mathrm{Res}_t(g_1, g')\mathrm{Res}_t(g_2, g')$. Now, let $\rho_1, \ldots, \rho_n$ be the roots of g_1 in the algebraic closure of U. Then

$$\begin{aligned}\mathrm{Res}_t(g_1, g') &= \mathrm{Lc}(g_1)^{\deg(g')} \prod_{i=1}^{n} (g_1 g_2)'(\rho_i) = \mathrm{Lc}(g_1)^{\deg(g')} \prod_{i=1}^{n} g_1'(\rho_i) g_2(\rho_i) \\ &= \mathrm{Lc}(g_1)^{\deg(g')-\deg(g_1' g_2)} \mathrm{Res}_t(g_1, g_1' g_2) = \mathrm{Res}_t(g_1, g_1')\mathrm{Res}_t(g_1, g_2).\end{aligned}$$

Repeating the reasoning for $\mathrm{Res}_t(g_2, g')$ one concludes the result. □

For the next lemma, we recall that f is the defining polynomial of $\mathcal{C}$.

Lemma 7 *The square-free part of $\mathrm{Res}_t(\Sigma_1, \Sigma_2)$ factors either as $f(a, b)$ or as $f(a, b)$ and some linear factors.*

Proof $\alpha_1^2 \Sigma_1 = \tilde{\Sigma}_1 = (p_1 + \sqrt{-1} p_2) - (a + \sqrt{-1} b) p$. The polynomial α_1^2 is a factor of the content of $\tilde{\Sigma}_1$ w.r.t. $\{a, b\}$, i.e. of $\gcd(p_1 + \sqrt{-1} p_2, p)$. Therefore, $p_1 + \sqrt{-1} p_2 = \alpha_1^2 \xi_1$ and $p = \alpha_1^2 \gamma_1$ for some $\xi_1, \gamma_1 \in \mathbb{K}[t]$. Thus, $\Sigma_1 = \xi_1 - (a + \sqrt{-1} b)\gamma_1$. A similar reasoning shows that $\Sigma_2 = \xi_2 - (a - \sqrt{-1} b)\gamma_2$ for some $\xi_2, \gamma_2 \in \mathbb{K}[t]$. Now, let $\theta_i = \gcd(\xi_i, \gamma_i)$, and $\tilde{\xi}_i, \tilde{\gamma}_i$ such that $\xi_i = \theta_i \tilde{\xi}_i$ and $\gamma_i = \theta_i \tilde{\gamma}_i$. We show that $\tilde{\xi}_1/\tilde{\gamma}_1$ is not constant. If $\tilde{\xi}_1/\tilde{\gamma}_1 \in \mathbb{K}$ then $\xi_1/\gamma_1 \in \mathbb{K}$, and hence $(p_1 + \sqrt{-1} p_2)/p \in \mathbb{K}$. But this implies that $\mathcal{C}$ is a line, and we have excluded this case.

Let $M_1 = \tilde{\xi}_1 - (a + \sqrt{-1} b)\tilde{\gamma}_1$ and $M_2 = \tilde{\xi}_2 - (a - \sqrt{-1} b)\tilde{\gamma}_2$, it holds that (up to multiplication by (-1))

$$\mathrm{Res}_t(\Sigma_1, \Sigma_2) = \mathrm{Res}_t(\theta_1, \theta_2)\mathrm{Res}_t(\theta_1, M_2)\mathrm{Res}_t(\theta_2, M_1)\mathrm{Res}_t(M_1, M_2).$$

$\mathrm{Res}_t(\theta_1, \theta_2) \in \mathbb{K}$. Moreover, by Lemmas 5 and 6, $\mathrm{Res}_t(\theta_1, \theta_2) \neq 0$. So, up to multiplication by non-zero elements in $\mathbb{K}$, it holds that

$$\mathrm{Res}_t(\Sigma_1, \Sigma_2) = \mathrm{Res}_t(\theta_1, M_2)\mathrm{Res}_t(\theta_2, M_1)\mathrm{Res}_t(M_1, M_2).$$

Furthemore, $\mathrm{Res}_t(\theta_1, M_2) \in \mathbb{K}[a - \sqrt{-1} b]$ and $\mathrm{Res}_t(\theta_2, M_1) \in \mathbb{K}[a + \sqrt{-1} b]$. So, each of these resultants is either constant or all its factors are linear in $\mathbb{K}[a, b]$. Let us finally see that the square-free part of $\mathrm{Res}_t(M_1, M_2)$ is $f(a, b)$. Since we have proved that $\tilde{\xi}_1/\tilde{\gamma}_1 \notin \mathbb{K}$, $(\tilde{\xi}_1/\tilde{\gamma}_1, \tilde{\xi}_2/\tilde{\gamma}_2)$ is a rational parametrization, by construction, in reduced form. Therefore, by Theorem 4.41 in [4], the square-free part of $\mathrm{Res}_t(M_1, M_2)$ is an irreducible polynomial.

On the other hand, since $\tilde{\Sigma}_i(p_1/p, p_2/p, t) = 0$, and $\alpha_i(t)^2\Sigma_i(a, b, t) = \tilde{\Sigma}_i(a, b, t)$, one has that $\Sigma_i(p_1/p, p_2/p, t) = 0$. Now, since $\mathrm{Res}_t(\Sigma_1, \Sigma_2) = M_1\Sigma_1 + M_2\Sigma_2$, for some $M_1, M_2 \in \mathbb{K}[a, b, t]$, one gets that $\mathrm{Res}_t(\Sigma_1, \Sigma_2)$ also vanishes at $\mathcal{P}(t)$. Therefore, $f(a, b)$ divides $\mathrm{Res}_t(\Sigma_1, \Sigma_2)$. Moreover, since $\mathcal{C}$ is not a line, then $f(a, b)$ divides $\mathrm{Res}_t(M_1, M_2)$. Finally, since f is irreducible and the square-part of $\mathrm{Res}_t(M_1, M_2)$ also, we conclude the proof. □

The next result shows the structure of $\mathcal{F}$

Proposition 1 *$\mathcal{F}$ decomposes as the union of $\mathcal{C}$ and finitely many lines.*

Proof By Lemma 6, $\mathcal{F}$ is defined by the square-free part of

$$\mathrm{Res}_t(\Sigma_1, \partial\Sigma_1/\partial t)\mathrm{Res}_t(\Sigma_2, \partial\Sigma_2/\partial t)\mathrm{Res}_t(\Sigma_1, \Sigma_2).$$

By Lemma 7, the square-free part of $\mathrm{Res}_t(\Sigma_1, \Sigma_2)$ factors as f or as f and some linear factors. Furthermore, since Σ_1 is linear in $a + \sqrt{-1}b$, $\mathrm{Res}_t(\Sigma_1, \partial\Sigma_1/\partial t)$ can be expressed as a polynomial in $(a \pm \sqrt{-1}\, b)$ and hence it is a product of linear factors in a, b. Similarly with $\mathrm{Res}_t(\Sigma_2, \partial\Sigma_2/\partial t)$. Therefore, all the other factors generate lines. □

Example 6 **(Double rational foci for the parabola)** Let $\mathcal{C}$ be the parabola over $\mathbb{C}$ parametrized by $\mathcal{P}(t) = (t, t^2)$. Using the above notation,

$$R(a, b) = -16(b - a^2)(4a + 4bi - i)(4a - 4bi + i).$$

Let $D_1(a, b) = 4a + 4bi - i$, and $D_2(a, b) = 4a - 4bi + i$. By Corollary 4, for $A \in \mathcal{C}$ the conchoid is rational. We analyze the lines given by D_1 and D_2. We take the parametrization $\mathcal{Q}(h) = (\frac{1}{4}i - h\,i, h)$ of D_1. Let $\Delta(t, h) = \|\mathcal{P}(t) - \mathcal{Q}(h)\|^2$. Then

$$\Delta(h, t) = \frac{1}{16}(4t^2 + 4it + 1 - 8h)(2t - i)^2.$$

The discriminant of $4t^2 + 4it + 1 - 8h$ w.r.t. t is $128(1 - 4h)$. So, the only candidate generated by D_1 is $\mathcal{Q}(1/4) = (0, \frac{1}{4})$ that, indeed, is a double rational focus. Analyzing D_2 one reaches the same point. So, the only double rational focus for the parabola $\mathcal{C}$ is $(0, \frac{1}{4})$ (see Example 10); note that we have got the focus of the parabola.

Example 7 **(Double rational foci for the circle)** Let $\mathcal{C}$ be the circle over $\mathbb{C}$ parametrized by

$$\mathcal{P}(t) = \left(\frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1}\right).$$

One has that

$$R(a, b) = 256(a^2 + b^2 - 1)(a + bi - i)(a + bi)(a - bi + i)(a - bi).$$

Therefore, $\mathcal{F}$ is the union of C the four lines defined $D_{1,1} := a + bi$, $D_{1,2} := a + bi - i$, $D_{2,1} := a - bi$, $D_{2,2} := a - bi + i$. By Corollary 4, we only need to analyze the lines given by $D_{i,j}$. We take the parametrization $\mathcal{Q}(h) = (-ih, h)$ of $D_{1,1}$. Let $\Delta(t,h) = \|\mathcal{P}(t) - \mathcal{Q}(h)\|^2$. Then

$$\Delta(h,t) = -(-t - i + 2th - 2ih)(t+i)(t-i)^2.$$

The resultant of $(-t-i+2th-2ih)(t+i)$ and its derivative w.r.t. t is $16h^2(-1+2h)$. So, the candidates generated by $D_{1,1}$ are $\mathcal{Q}(0) = (0,0)$ and $\mathcal{Q}(1/2) = \left(-\frac{i}{2}, \frac{1}{2}\right)$. One checks that $(0,0)$ is double rational but $\left(-\frac{i}{2}, \frac{1}{2}\right)$ is not. Reasoning with the other three lines no new foci are found. So, the only double rational focus is the center of the circle.

Example 8 **(Double rational foci for the ellipse)** Let C be the ellipse over $\mathbb{C}$ parametrized by

$$\mathcal{P}(t) = \left(\frac{4t}{t^2+1}, \frac{3(t^2-1)}{t^2+1}\right).$$

One has that

$$R(a,b) = (a^2/4 + b^2/9 - 1)D_{1,1}D_{1,2}D_{1,3}D_{2,1}D_{2,2}D_{2,3},$$

where $D_{1,1} = a + bi + \sqrt{5}i$, $D_{1,2} = a + bi - \sqrt{5}i$, $D_{1,3} = a + bi - 3i$, and $D_{2,j}$ is the conjugate polynomial of $D_{1,j}$. By Corollary 4, we only need to analyze the lines given by $D_{i,j}$. We take the parametrization $\mathcal{Q}(h) = (-ih - i\sqrt{5}, h)$ of $D_{1,1}$. Let $\Delta(t,h) = \|\mathcal{P}(t) - \mathcal{Q}(h)\|^2$. Then

$$\Delta(h,t) = \frac{-(\sqrt{5}+3)}{4}(-3t^2 - 4it + 3 + \sqrt{5} + 2ht^2 + 2h \\ + t^2\sqrt{5})(2t - 3i + i\sqrt{5})^2.$$

The resultant of $3t^2 - 4it + 3 + \sqrt{5} + 2ht^2 + 2h + t^2\sqrt{5}$ and its derivative w.r.t. t is

$$32\,(5/2 - 3/2\,h - 3/2\,\sqrt{5} + h^2 + 3/2\,h\sqrt{5})h.$$

So, the candidates generated by $D_{1,1}$ are (let $\alpha = 3/2 - 1/2\sqrt{5}$)

$$\mathcal{Q}(\alpha) = (-i\alpha - i\sqrt{5}, \alpha),\ \mathcal{Q}(-\sqrt{5}) = (0, -\sqrt{5}),\ \mathcal{Q}(0) = (-i\sqrt{5}, 0).$$

One checks that $\mathcal{Q}(-\sqrt{5})$ and $\mathcal{Q}(0)$ are double rational but $\mathcal{Q}(\alpha)$ is not. Using $D_{1,2}$ one deduces that $(0, \sqrt{5})$ and $(i\sqrt{5}, 0)$ are also double rational foci. Reasoning with the other lines no new foci are found. So, the only double rational focus are $\{(0, \pm\sqrt{5}), (\pm i\sqrt{5}, 0)\}$. Note that the foci of the ellipse have appeared.

Example 9 (Double rational foci for the hyperbola) Let $\mathcal{C}$ be the hyperbola over $\mathbb{C}$ parametrized by

$$\mathcal{P}(t) = \left(\frac{-1-t^2}{-1+t^2}, \frac{2t}{-1+t^2} \right).$$

One has that

$$R(a, b) = (a^2 - b^2 - 1) D_{1,1} D_{1,2} D_{1,3} D_{2,1} D_{2,2} D_{2,3},$$

where $D_{1,1} = a + bi - \sqrt{2}$, $D_{1,2} = a + bi + \sqrt{2}$, $D_{1,3} = a + bi + 1$, and $D_{2,j}$ is the conjugate polynomial of $D_{1,j}$. Reasoning as before, one deduces that the double rational foci are $(\pm\sqrt{2}, 0)$, $(0, \pm i\sqrt{2})$. Note that the foci of the hyperbola have appeared.

It is clear that the center of a circle is a double rational focus. Moreover the above examples show that the foci of an specific ellipse, hyperbola and parabola are also double rational focus. Furthermore, it is also clear that there are rational double foci different of the conic foci. In the following theorem, we prove that the above behavior of the conic foci is not specific of the previous examples; see also Example 6.2. in [1] for an alternative reasoning to derive this result.

Theorem 6 *A focus of a conic is always a rational double focus.*

Proof Let $\mathcal{C}$ be an ellipse, a hyperbola or a parabola defined by a real polynomial. We can assume w.l.o.g. that the coordinate system is such that $\mathcal{C}$ is given by its reduced equation $f(y_1, y_2)$. So, if $\mathcal{C}$ is a ellipse, then $f(y_1, y_2) = y_1^2/\alpha^2 + y_2^2/\beta^2 - 1$. Let $A^{\pm} = (\pm c, 0)$ be the foci of the ellipse, then $\alpha^2 = \beta^2 + c^2$. We consider the proper parametrization $\mathcal{P}(t) = (2\alpha t/(t^2+1), \beta(t^2-1)/(t^2+1))$ of $\mathcal{C}$, and we show that $\mathcal{P}(t)$ is $A^{\pm}$–RDF. We prove it for A^+; similarly for A^-.

$$\|\mathcal{P}(t) - A^+\|^2 = \frac{(\beta^2 + c^2) - 4\alpha c t + (-2\beta^2 + 4\alpha^2 + 2c^2)t^2 - 4\alpha c t^3 + (\beta^2 + c^2)t^4}{(t^2+1)^2}$$

and using that $\alpha^2 = \beta^2 + c^2$ one gets that

$$\|\mathcal{P}(t) - A^+\|^2 = \frac{(-2ct + \alpha + \alpha t^2)^2}{(t^2+1)^2}.$$

If $\mathcal{C}$ is a hyperbola, then $f(y_1, y_2) = y_1^2/\alpha^2 - y_2^2/\beta^2 - 1$. Let $A^{\pm} = (\pm c, 0)$ be the foci of the hyperbola, then $c^2 = \alpha^2 + \beta^2$. We consider the proper parametrization $\mathcal{P}(t) = (2\alpha t/(t^2+1), \beta\sqrt{-1}(t^2-1)/(t^2+1))$ of $\mathcal{C}$. Reasoning as above, one gets that $\mathcal{P}(t)$ is $A^{\pm}$–RDF.
Finally, let $\mathcal{C}$ be a parabola. Then, $f(y_1, y_2) = y_2^2 - 2\lambda y_1$, and the focus $A = \left(\frac{\lambda}{2}, 0\right)$. Then, considering the proper parametrization $\mathcal{P}(t) = (t^2/(2\lambda), t)$. Now $\|\mathcal{P}(t) - A\|^2 = \frac{(t^2+\lambda^2)^2}{4\lambda^2}$. □

Remark 6 Repeating the computations of Examples 6,7, 8, 9 for the conics, in reduced form, we get that the double rational foci are

1. for the parabola $y_2^2 = 2\lambda y_1$, one gets $\left(\frac{\lambda}{2}, 0\right)$,
2. for the circle, its center,
3. for the ellipse $y_1^2/\alpha^2 + y_2^2/\beta^2 = 1$, one gets $(\pm c, 0)$ and $(0, \pm\sqrt{-1}c)$, with $\alpha^2 = \beta^2 + c^2$,
4. for the hyperbola $y_1^2/\alpha^2 - y_2^2/\beta^2 = 1$, one gets $(\pm c, 0)$ and $(0, \pm\sqrt{-1}c)$, with $c^2 = \alpha^2 + \beta^2$.

As additional information, we give here the defining polynomial of the corresponding set $\mathcal{F}$.

(i) $(2a - \lambda + 2\sqrt{-1}b)(-2a + \lambda + 2\sqrt{-1}b)(2a\lambda - b^2)$ for the parabola.
(ii) $(a - \sqrt{-1}b - c)(c + b\sqrt{-1} + a)(c - \sqrt{-1}b + a)(a + b\sqrt{-1} - c)(-\sqrt{-1}b + a + \beta\sqrt{-1})(b\sqrt{-1} + a - \sqrt{-1}\beta)(a^2\beta^2 - \beta^2c^2 - \beta^4 + b^2c^2 + \beta^2b^2)$ for the ellipse.
(iii) $(a + b\sqrt{-1} + c)(-c + b\sqrt{-1} + a)(-c - \sqrt{-1}b + a)(a - \sqrt{-1}b + c)(\beta + a + b\sqrt{-1})(a - \beta - \sqrt{-1}b)(\beta^2a^2 - \beta^2c^2 + \beta^4 - b^2c^2 + \beta^2b^2)$ for the hyperbola.

Detecting rational foci. For analyzing the general case, we apply Theorem 5. Therefore, we study the genus of $\mathfrak{G}(\mathcal{P})$ in terms of the parameters a, b that define the focus; recall that $\mathcal{P}(t)$ is a proper parametrization of $\mathcal{C}$. In the sequel we see that this can be carried out algorithmically.

First, we need to compute the primitive part w.r.t. x_2 of the numerator of $\mathfrak{b}((-2x_2, x_2^2 - 1), \mathcal{P}(x_1) - A)$. That is, if Δ_i are as in the previous subsection, we need to compute the primitive part w.r.t. x_2 of

$$g(x_1, x_2, a, b) := -2x_2\Delta_1(a, x_1) + (x_2^2 - 1)\Delta_2(b, x_1).$$

The content of g w.r.t. x_2 is $\gcd(\Delta_1, \Delta_2)$. If none of the leading coefficients w.r.t. x_1 of Δ_1 and Δ_2 is constant, we analyze (as a particular case) the focus A obtained by solving, in $\{a, b\}$, the linear system $\{\mathrm{Lc}_{x_1}(\Delta_1(a, x_1)) = 0, \mathrm{Lc}_{x_1}(\Delta_2(b, x_1)) = 0\}$. So, let us assume, in the sequel, that at least one of these leading coefficients is constant. Let $S(a, b) = \mathrm{Res}_{x_1}(\Delta_1, \Delta_2)$, and let $\mathcal{H}$ be the algebraic set defined by S over $\mathbb{K}$. Then,

Proposition 2 *$g(x_1, x_2, a_0, b_0)$ is primitive w.r.t. x_2 iff $(a_0, b_0) \notin \mathcal{H}$.*

In addition, we observe that if $g(x_1, x_2, a_0, b_0)$ is primitive w.r.t. x_2, since $\mathcal{C}$ is not a line, then $g(x_1, x_2, a_0, b_0)$ is square-free (see Remark 2). In the following proposition, we see the structure of $\mathcal{H}$.

Proposition 3 *$\mathcal{H}$ is either $\mathcal{C}$ or it is the union of $\mathcal{C}$ and finitely many lines.*

Proof Let $\alpha_1(x_1)$ be the content of Δ_1 w.r.t. a, $\alpha_2(x_1)$ the content of Δ_2 w.r.t. b, and let $\tilde{\Delta}_i$ the corresponding cofactor of α_i in Δ_i. Then, S is $\mathrm{Res}_{x_1}(\alpha_1, \alpha_2)\mathrm{Res}_{x_1}(\alpha_1, \tilde{\Delta}_2)$ $\mathrm{Res}_{x_1}(\alpha_2, \tilde{\Delta}_1)\mathrm{Res}_{x_1}(\tilde{\Delta}_1, \tilde{\Delta}_2)$. Now, observe that $\mathrm{Res}_{x_1}(\alpha_1, \alpha_2) \in \mathbb{K}$, $\mathrm{Res}_{x_1}(\alpha_1, \tilde{\Delta}_2) \in$

$\mathbb{K}[b]$, $\mathrm{Res}_{x_1}(\alpha_2, \tilde{\Delta}_1) \in \mathbb{K}[a]$ and that $\mathrm{Res}_{x_1}(\tilde{\Delta}_1, \tilde{\Delta}_2) = f(a, b)$ (see Theorem 4.41 in [4] and recall that $\mathcal{P}(t)$ is proper). The proof ends observing that none of the three first resultants is zero, because $\gcd(p_1, p_2, p) = 1$. □

Now, the strategy consists in looking for rational foci separately in $\mathcal{H}$ and in $\mathbb{K}^2 \setminus \mathcal{H}$.

[Rational foci in $\mathcal{H}$] By Proposition 3, all components of $\mathcal{H}$ are rational. For each component of $\mathcal{H}$ we consider a proper normal (i.e. surjective) parametrization $\mathcal{Q}(t) = (\xi_1(t), \xi_2(t))$; see Theorem 6.26 in [4]. Note that we can assume w.l.o.g that all coefficients in $\mathcal{Q}(t)$ are in a finite algebraic extension of the computable subfield $\mathbb{L}$. This is because $\alpha_i, \tilde{\Delta}_i \in \mathbb{L}[a, b][x_1]$, and because of Corollary 5.9. in [4]. Then, we take the primitive part of $g(x_1, x_2, \xi_1(t), \xi_2(t)) \in \mathbb{L}(t)[x_1, x_2]$ w.r.t. x_2. This implies computing gcds in the Euclidean domain $\mathbb{L}(t)[x_1]$. During these gcd computations, zero-tests in $\mathbb{L}(t)$ are required, and this might imply to analyze separately some particular values of the parameter t; and hence some particular foci. Let $h(x_1, x_2, t)$ be the primitive part of $g(x_1, x_2, \xi_1(t), \xi_2(t))$ w.r.t. x_2. Additionally, note that, because of Lemma 2, the double rational foci analysis provides the foci for which $\mathfrak{G}(\mathcal{P})$ is reducible. So, we can assume w.l.o.g. that h is irreducible over the algebraic closure of $\mathbb{L}(t)$.

Finally, we are ready to analyze the genus of the curve $\mathfrak{G}(\mathcal{P})$ defined by h, and depending on the parameter t. For this purpose we show that the algorithm described in Sect. 3.3 in [4], based on the use of conjugate families of points, can be applied in our case. This algorithm essentially uses: resultant computations, gcd computations, and factorization of polynomials over simple algebraic extensions of $\mathbb{L}(t)$. Therefore it can be performed algorithmically. Of course, particular values of the parameter t might need to be analyze from the zero-tests appearing in the gcds, leading coefficients of polynomials, and detection of the character of the conjugate families of singularities.

[Rational foci in $\mathbb{K}^2 \setminus \mathcal{H}$] By Proposition 2, we know that $g(x_1, x_2, a, b)$ is primitive w.r.t. x_2. Moreover, a, b can be treated as transcendental elements and, as commented before, because of Lemma 2, we can assume w.l.o.g. that g is irreducible in $\mathbb{K}[a, b, x_1, x_2]$. Then, we analyze the genus of the curve $\mathfrak{G}(\mathcal{P})$ defined by g, and depending on a, b. As above, we base our reasoning in the algorithm described in Sect. 3.3 in [4]. Resultant computations, gcd computations, and factorization of polynomials over simple algebraic extensions of $\mathbb{L}(a, b)$ do not present complications. However, when dealing with zero-tests during these computations, we might have to introduce special subcases for particular values of a, b. These special subcases will be given by a proper algebraic set in $\mathbb{K}^2$. If its dimension is zero, finitely many foci need to be treated. If the dimension is 1, we take each irreducible component (say $M(a, b)$ is its defining polynomial) and we work in $\mathbb{F}[x_1, x_2]$, where $\mathbb{F}$ is the quotient field of $\mathbb{L}[a, b]/(M(a, b))$; i.e. $\mathbb{F}$ is the field of rational functions over $\mathbb{L}$ of the corresponding curve. Clearly, gcds and resultants can be performed in $\mathbb{F}[x_1, x_2]$. For factoring univariate polynomials over simple algebraic extensions of $\mathbb{F}$, we need to show that we can factor univariate polynomials over $\mathbb{F}$. However, $\mathbb{F}$ can be seen as $\mathbb{L}(a)[b]/(M(a, b))$ where a is seen as transcendental over $\mathbb{L}$, and b algebraic over $\mathbb{L}(a)$ being M its minimal polynomial over $\mathbb{L}(a)$. Then, factorization in $\mathbb{F}[z]$ can be carried out. Of course, zero-tests may generate again special subcases, but now with finitely many candidates. Therefore, also in this case, the analysis can be performed algorithmically.

Example 10 **(Rational foci for the parabola)** Let $\mathcal{C}$ be the parabola over $\mathbb{C}$ parametrized by $\mathcal{P}(t) = (t, t^2)$. Then, using the notation above,

$$g(x_1, x_2, a, b) = -2x_2x_1 + 2x_2a + x_2^2x_1^2 - x_2^2b - x_1^2 + b,$$

and $\mathcal{H} = \mathcal{C}$. We also know, by Lemma 2 and Example 6, that the primitive part w.r.t. x_2 of g is irreducible iff $A \neq (0, 1/4)$.

[Case $\mathcal{H}$] By Corollary 4, every focus on $\mathcal{C}$ is rational. So, we do not need a further analysis. Nevertheless, in order to illustrate the method, we proceed as discussed above. Then $h = -tx_2^2 + t + 2x_2 - x_1x_2^2 + x_1$ for all values of t. We study the genus. The curve has a singularity at $(1 : 0 : 0)$ whose multiplicity is 2, independently of the values of t. On the other hand, $\gcd(\mathrm{Res}_{x_2}(h, \partial h/\partial x_1), \mathrm{Res}_{x_2}(h, \partial h/\partial x_2)) = 1$ independently of the values of t. So, for all t values, none additional singularities appear. Since the curve has degree 3, all points on $\mathcal{C}$ are rational foci.

[Case $\mathbb{C}^2 \setminus \mathcal{H}$] In this case, we work with $g(x_1, x_2, a, b)$. The points at infinity $P_1 := (1 : 0 : 0)$ and $P_2 := (0 : 1 : 0)$ are, independently on a, b, double points. Moreover, P_1 is always ordinary (the tangents are given by $x_2^2 - x_3^2$) and, if $b \neq 0$, P_2 is ordinary too (the tangents are given by $x_1^2 - bx_3^2$).

Now, we analyze the affine singular locus. For this purpose, we compute first $\gcd(\mathrm{Res}_{x_2}(g, \partial g/\partial x_1), \mathrm{Res}_{x_2}(g, \partial g/\partial x_2))$. This gcd is 1 in the open set $\Omega = \{(a, b) \in \mathbb{C}^2 \mid (-b+a^2)b^2(16b^2-8b+1+16a^2) = 0\}$. This set decomposes as $\mathcal{C} \cup \mathcal{L}_1 \cup \mathcal{L}_2^+ \cup \mathcal{L}_2^-$, where $\mathcal{L}_1$ is the line defined by $b = 0$, and $\mathcal{L}_2^{\pm}$ the lines defined by $a \pm \left(\frac{1}{4} - b\right)\sqrt{-1} = 0$. Since the case of $(a, b) \in \mathcal{C}$ has been already treated, we distinguish the following subcases. For $(a, b) \in \mathbb{C}^2 \setminus \mathcal{H}$:

1. If $(a, b) \notin \mathcal{L}_1 \cup \mathcal{L}_2^+ \cup \mathcal{L}_2^-$, then $\mathfrak{G}(\mathcal{P})$ has no affine singularities and the singularities at infinity are ordinary. Therefore, the genus is 1 and none rational focus appears.
2. Let $(a, b) \notin \mathcal{L}_1$; i.e. $b \neq a^2$ and $b \neq 0$. Now,
   2.1. Let $(a, b) \in \mathcal{L}_2^+$. Then, $A = ((1/4-b)\sqrt{-1}, b)$. In this case, we work over $\mathbb{Q}(\sqrt{-1})(b)$, and the new defining polynomial of $\mathfrak{G}(\mathcal{P})$ is $g_1(x_1, x_2, b) = g(x_1, x_2, (1/4 - b)\sqrt{-1}, b)$. Now, when performing the computations of the gcd of $\mathrm{Res}_{x_2}(g_1, \partial g_1/\partial x_1)$, and $\mathrm{Res}_{x_2}(g_1, \partial g_1/\partial x_2)$, new particular subcases appears. Namely,
      2.1.1. If $b \notin \{\pm 1/4\}$ the above gcd is $(\sqrt{-1} - 2x_1)^2$. Moreover, the gcd of g_1, $(\partial g_1/\partial x_1)$, $(\partial g_1/\partial x_2)$, evaluated at $(\sqrt{-1}/2, x_2, b)$, is $(x_2 + \sqrt{-1})$. So, $(\sqrt{-1}/2, -\sqrt{-1})$ is an affine singularity. Moreover, the second derivatives do not all vanish at $(\sqrt{-1}/2, -\sqrt{-1})$, and hence it is a double point. Therefore, the genus is 0. Thus, all the foci in this subcase are rational.
      2.1.2. Let $b \in \{\pm 1/4\}$. If $b = 1/4$ we get $(0, 1/4)$ that has been excluded because of being a double rational focus. If $b = -1/4$ we get $(1/2\sqrt{-1}, -1/4)$ that has been excluded because it is on the parabola.
   2.2. Let $(a, b) \in \mathcal{L}_2^-$. Then a similar process as in (2.1.) provides that for $b \notin \{\pm 1/4\}$ the genus is zero, and hence rational foci are provided; that

for $b = 1/4$ one gets the excluded focus $(0, 1/4)$; and that for $b = -1/4$ one gets the focus $(-1/2\sqrt{-1}, -1/4)$ on the parabola.

3. Let $(a, b) \in \mathcal{L}_1$; i.e. $b \neq a^2$ and $b = 0$. In this case, we work over $\mathbb{Q}(a)$, and the new defining polynomial of $\mathfrak{G}(\mathcal{P})$ is $g_2(x_1, x_2, a) = g(x_1, x_2, a, 0)$. Now, when performing the computations of the gcd of $\mathrm{Res}_{x_2}(g_2, \partial g_2/\partial x_2)$, and $\mathrm{Res}_{x_2}(g_2, \partial g_2/\partial x_2)$, new particular subcases appears. Namely,
   3.1. Let $a \notin \{\pm 3/8\sqrt{-1}, \pm\sqrt{11}/8, \pm 1/2, \pm\sqrt{-1}/4\}$. Then, the gcd is 1, and no affine singularity appear. So, the singular locus is $\{(1:0:0), (0:1:0)\}$ being both double points. Although $(1:0:0)$ is ordinary, $(0:1:0)$ is not, and hence we need to blow it up. For this purpose, we work with the homogenization of g_2. We apply a suitable projective linear change of coordinates such that no line $x_i = 0$ is tangent to the curve at $(0:1:0)$, for instance $\{x_1 = x_1^* - x_3^*, x_2 = x_2^* - x_3^*, x_3 = x_1^* + x_3^*\}$, and the Cremona transformation. Then, one deduces that, for $a \neq 0$ (which is our case), $(0:1:0)$ has none neighboring singularities. So, in this case the genus of $\mathfrak{G}(\mathcal{P})$ is 1, and thus no new rational foci appear.
   3.2. For $a \in \{\pm 3/8\sqrt{-1}, \pm\sqrt{11}/8, \pm 1/2\}$ one gets genus 1, and hence no additional rational focus. For $a \in \{\pm\sqrt{-1}/4\}$ one gets genus 0. These two last foci correspond to $\mathcal{L} \cap \mathcal{L}_2^+$ and $\mathcal{L} \cap \mathcal{L}_2^-$.

Summarizing, jointly with Example 6, one has the following table

	Double rational foci	Rational foci
Parabola (t, t^2)	$\left(0, \frac{1}{4}\right)$	$(a, a^2),\ a \in \mathbb{C}$ $\left(\pm\left(\frac{1}{4} - b\right) i, b\right)\right),\ b \in \mathbb{C} \setminus \{\frac{1}{4}\}$

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